week 7 :

Recall : $V \subseteq \mathbb{R}^n$ is a subspace if

① $V \neq \phi$

② if $x, y \in V$, then $x+y \in V$

③ if $x \in V$, $\alpha \in \mathbb{R}$ then $\alpha x \in V$.

~~**~~ Column space of a $m \times n$ matrix A

$C(A) = \{ y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ s.t. } Ax = y \}$ is a subspace of $\mathbb{R}^m$.

And if $A = [A_1 | A_2 | \cdots | A_n]$ where A_i is $m \times 1$ matrix

then $C(A) = \text{span}\{A_1, \cdots, A_n\}$. ($\underline{\text{Recall}}$ is a subspace)

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proof of $C(A) = \text{span}\{A_1, \cdots, A_n\}$

($\Rightarrow$): If $y \in C(A)$, then $\exists x \in \mathbb{R}^n$ s.t. $Ax = y$

write $A = [A_1, \cdots, A_n]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ yields

$Ax = A_1 x_1 + \cdots + A_n x_n = y \in \mathbb{R}^m$.

$\therefore y \in \text{span}\{A_1, \cdots, A_n\}$

($\Leftarrow$) : If $y \in \text{span}\{A_1, \cdots, A_n\}$,

$y = \sum_{i=1}^{n} x_i A_i$ for some $x_i \in \mathbb{R}$

$= [A_1 \cdots A_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax$. where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ #

RmK: vector subspace is closed under linear combination.

i.e. if $V \subseteq \mathbb{R}^n$ is a subspace,

then linear combination of elements in V is in V.

(By defn).

Recall (the Q):

Q: Given $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$,

ask if $\begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} \in \operatorname{span}(S)$.

Method: by solving the system of linear equ.

, find $x_1, x_2, x_3 \in \mathbb{R}$ st.

$$\begin{cases} 1 \cdot x_1 + (-1)\, x_2 + 2 \cdot x_3 = 1 \\ 2 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = 8 \\ 1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = 5 \end{cases}$$

Consider
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow{\text{Row op.}} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF.}}$$

$\therefore$ The system is consistent w/ ∞ many sol.

$$\text{sol.} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-t \\ 2+t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

take $t=0$ for example

$$\Rightarrow \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\in \operatorname{span}(S)$. #

Thm Given $S = \{u_1, u_2, \cdots, u_n\} \subseteq \mathbb{R}^n$

Denote $A = [u_1 | u_2 | \cdots | u_n]$, the $n \times n$ matrix,

then A is non-singular iff $\langle S \rangle = \mathbb{R}^n$

pf: $(\Rightarrow)$ $\forall b \in \mathbb{R}^n$, $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$

$\Rightarrow \langle S \rangle = C(A) = \mathbb{R}^n$

$(\Leftarrow)$ if $\langle S \rangle = \mathbb{R}^n$.

$\Rightarrow \forall b \in \mathbb{R}^n$, $\exists x_1 \ldots x_n \in \mathbb{R}$ s.t.

vector coeff.

$\sum_{i=1}^{n} u_i x_i = b$.

$\parallel$

$Ax = b$. where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

i.e. $\mathcal{LS}(A, b)$ has at least 1 sol, $\forall b \in \mathbb{R}^n$

$\Rightarrow A$ is invertible #

More common example of subspace

(a) Let $A = m \times n$ matrix, Z = subspace of $\mathbb{R}^n$.

then $W = \{x \in \mathbb{R}^n \mid Ax \in Z\}$ is a subspace of $\mathbb{R}^n$.

"pre-image of Z under A ($x \mapsto Ax$)".

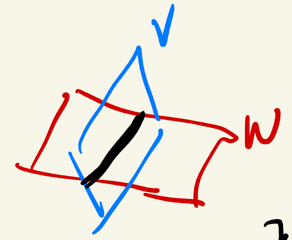
If $Z = \{0\}$, then $W = N(A)$.

(b) Let $H = p \times q$ matrix, Z = subspace of $\mathbb{R}^q$

$W = \{ y \in \mathbb{R}^p \mid \exists u \in Z \text{ s.t. } y = Hu \}$ "Image of Z under A"

If $Z = \mathbb{R}^q$, then $W = C(A)$.

ⓒ If V, W are subspace of $\mathbb{R}^n$, eg:



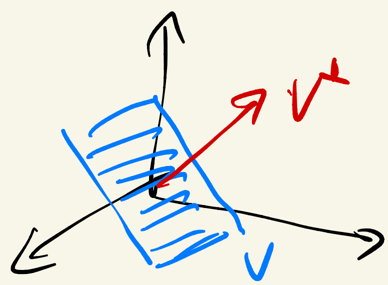
then $V \cap W$ = subspace = $\{x \in \mathbb{R}^n \mid x \in V, \text{ and } x \in W\}$

ⓓ If V, W are subspace, then

$V + W = \{x \in \mathbb{R}^n \mid x = v + w \text{ for some } v \in V, w \in W\}$

ⓔ If V is a subspace in $\mathbb{R}^n$

$V^\perp = \{x \in \mathbb{R}^n \mid x^t y = 0 \ \forall y \in V\}$



<u>Question</u>: span of fewer elements ??

<u>Example</u>: $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

<u>Q</u>: ask if span S = span $\{v_1, v_2, v_3\}$?? (3 elements)

random target

Consider $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & a \\ 2 & 1 & 1 & 1 & b \\ 1 & 1 & 0 & 1 & c \end{array}\right]$

$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & b-c \\ 0 & 1 & -1 & 0 & \frac{c-a}{2} \\ 0 & 0 & 0 & 1 & -b + \frac{a}{2} + \frac{3}{2}c \end{array}\right]$

$$\therefore \text{sol. set} = \left\{ \begin{bmatrix} b-c-t \\ \frac{c-a}{2}+t \\ t \\ -b+\frac{a}{2}+\frac{3}{2}c \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{bmatrix} -c+b \\ \frac{c-a}{2} \\ 0 \\ -b+\frac{a}{2}+\frac{3}{2}c \end{bmatrix}}_{\text{depends on target}} + t \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{corresponds to Null space "}Ax=0\text{"}} : t \in \mathbb{R} \right\}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (b-c-t)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{c-a}{2}+t\right)\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$+ t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \left(-b+\frac{a}{2}+\frac{3}{2}c\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ If taking $t=0$ all the time,

then we can still conclude $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{span}\{v_1, v_2, v_4\}$.

$\therefore \text{span}\{v_1, v_2, v_4\} = \mathbb{R}^3$

From above: Increase coeff. of v_3 by 1

$\Downarrow$

increase coeff. of v_1 by -1

of v_2 by 1

i.e. $\cancel{t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -\left(\cancel{t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (-\cancel{t})\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) \Rightarrow v_3 = v_1 - v_2$

depends on v_1, v_2.

$\in \text{span}\{v_1, v_2\}$.

In general, need a concept to Rule out $v_3 \in \operatorname{span}\{v_1, v_2\}$

<u>Defn</u>: $S = \{u_1, u_2, \ldots, u_n\}$ is said to be linearly independent if the following is true:

the only $\{\lambda_i\}_{i=1}^{n}$ s.t. $\sum_{i=1}^{n} \lambda_i u_i = 0$ is $\{0, 0, \cdots, 0\}$.

<u>Eg</u>: $S = \left\{ \underset{= v_1}{\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}}, \underset{= u_2}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}}, \underset{= u_3}{\begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}}, \underset{= v_4}{\begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

is linearly dep.

<u>checking</u>: consider $[A \mid 0]$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 2 & -6 & 0 \\ -1 & 2 & 1 & 7 & 0 \\ 3 & -1 & -3 & -1 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row op.}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ sol. set of $LS(A, 0)$ $= \left\{ t \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} \left(= \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix} \right\} \right)$

$\therefore$ $Ax = 0$ for some $x \neq 0 \in \mathbb{R}^4$.

Eg:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$= v_1, \; = v_2, \; = v_3, \; = v_4$

is linearly indep.

checking: consider $[A|0]$

$$\left[\begin{array}{cccc|c} 2 & 1 & 2 & -6 & 0 \\ -1 & 2 & 1 & 7 & 0 \\ 3 & 4 & -3 & -1 & 0 \\ 1 & 5 & 6 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row op.}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Sol. set} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Null}(A).$$

unique sol. !! ∦

generalize

Thm: Let $S = \{v_1, \cdots, v_n\} \subseteq \mathbb{R}^m$ and $A = [v_1 | v_2 | \cdots | v_n]$

be $m \times n$ matrix. Then S is linearly indep iff

$LS(A,0)$ has a unique solution.

pf: $(\Leftarrow)$ $LS(A,0)$ has unique sol. $\Rightarrow$ sol. $= \vec{0}$.

$\therefore$ There is No non-trivial sol. to $Ax=0$

$\sum_{i=1}^{n} \lambda_i v_i = 0$

$(\Rightarrow)$ If S is linearly indep.

Consider $Ax=0 \iff \sum_{i=1}^{n} x_i V_i = 0$ $\left(x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)$

assumption $\Rightarrow$ $x_i = 0 \quad \forall i = 1, 2, \cdots, n$

$\Rightarrow N(A) = \{0\}$

$\Rightarrow LS(A,0) = \{0\}$ #.

Lemma (linear map "preserve" the linear independence)

Let $\{u_1, u_2, \cdots, u_n\} \in \mathbb{R}^m$, $A = n \times m$ matrix

① If $\{Au_1, Au_2, \cdots, Au_n\}$ is linearly indep

then so does $\{u_1, \cdots, u_n\}$

② If A = non-singular, and $\{u_1, \cdots, u_n\}$ is linearly indep. then so does $\{Au_1, \cdots, Au_n\}$.

pf: ①: Let α_i be s.t.

$$\sum_{i=1}^{n} \alpha_i u_i = 0 \Rightarrow \sum_{i=1}^{n} \alpha_i (Au_i) = 0$$

assumption $\Rightarrow$ $\alpha_i = 0 \quad \forall i = 1, 2, \cdots, n$ #

②: Let α_i be s.t. $\sum_{i=1}^{n} \alpha_i A u_i = 0$

$$\Rightarrow A^{-1}\left(\sum_{i=1}^{n} \alpha_i (A u_i)\right) = \sum_{i=1}^{n} \alpha_i u_i = 0$$

$$\Rightarrow \alpha_i = 0 \quad \forall i = 1, 2, \cdots n$$ #

Q: What is the geometric meaning of solving LS(A, 0) to determine linearly indep??

Linear dependence: $\exists \{\alpha_i\}_{i=1}^{n}$ which is NOT constantly zero s.t. $\sum_{i=1}^{n} \alpha_i V_i = 0$

$\Rightarrow$ there is at least $\alpha_{i_0} \neq 0$

$$\Rightarrow V_{i_0} = \frac{-1}{\alpha_{i_0}} \sum_{\substack{j=1 \\ j \neq i_0}}^{n} \alpha_j V_j \in \operatorname{span}\{V_i \mid i \neq i_0\}$$

Eg: $S = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -1 \\ 19 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -1 \\ 6 \end{bmatrix} \right\}$

$$[A|0] \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} = RREF$$

$$\therefore \text{sol. set of } LS(A,0) = \left\{ \begin{bmatrix} -s-t \\ -s-t \\ s-t \\ -t \\ t \\ s \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

i.e. $\begin{cases} -V_1 - V_2 + V_3 + V_6 = 0 \\ -V_1 - V_2 - V_3 - V_4 + V_5 = 0 \end{cases}$

$\Rightarrow V_6, V_5 \in \text{Span}\{V_1, V_2, V_3, V_4\}$

corresponds to the free column!!

Thm: If $S = \{u_1, \dots, u_m\} \subseteq \mathbb{R}^n$ with $m > n$,

then S must be linearly dependent.

pf: $A \to B = \text{RREF}$.

$\because m > n \quad \therefore$ There must be free column in B.

$\Rightarrow$ $Ax = 0$ admits non-trivial sol.

$\Rightarrow$ linear dependent.

Thm: Given a square matrix A,

A is non-singular iff $C(A)$ is linearly indep.

pf:

non-singular $\Leftrightarrow$ $\mathcal{LS}(A, 0) = \{0\}$

$\Leftrightarrow$ column vectors of A are linearly independent.

Eg: find $N(A)$ where (in term of span)

$$A = \begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}$$

$$A \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}.$$

free column

$\therefore$ x_4, x_5 = free variables.

① taking $x_4 = 1$, $x_5 = 0$ $\Rightarrow$ $\begin{bmatrix} x_1 = -1 \\ x_2 = 2 \\ x_3 = -2 \\ x_4 = 1 \\ x_5 = 0 \end{bmatrix}$ B a sol. to $Ax=0$

② taking $x_4 = 0$, $x_5 = 1$, $\begin{bmatrix} x_1 = 2 \\ x_2 = -2 \\ x_3 = 1 \\ x_4 = 0 \\ x_5 = 1 \end{bmatrix}$ B another sol. to $Ax=0$

$$\therefore \text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ why?? #